Introduction to Celestial Mechanics

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1 Kepler's Laws

To study the orbits and the motion of objects under gravitational force, the natural introduction is the three laws devised by Kepler to explain the planetary orbits.

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

2 Two Body Problem

The main problem in celestial mechanics is that of two bodies bound gravitationally, in orbit about each other. Consider two bodies of masses m_1 and m_2 , with radius vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ with respect to a coordinate system fixed at their centre of mass as origin. Then we have:

$$\frac{m_1\mathbf{r_1} + m_1\mathbf{r_2}}{m_1 + m_2} = 0$$

Define $\mathbf{r} = \mathbf{r_2} - \mathbf{r_1}$, the relative position vector, then:

$$\mathbf{r_1} = -\frac{m_2}{m_1 + m_2}\mathbf{r};\ \mathbf{r_2} = \frac{m_1}{m_1 + m_2}\mathbf{r}$$

Define $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$, then:

$$\mathbf{r_1} = -\frac{\mu}{m_1}\mathbf{r};\ \mathbf{r_2} = \frac{\mu}{m_2}\mathbf{r}$$

Newton's law of gravitation for the two bodies can be written as :

$$m_1\ddot{\mathbf{r_1}} = -G\frac{m_1 m_2}{r^3}(\mathbf{r_1} - \mathbf{r_2})$$

$$m_2\ddot{\mathbf{r_2}} = -G\frac{m_1 m_2}{r^3}(\mathbf{r_2} - \mathbf{r_1})$$

$$\Rightarrow m_1\ddot{\mathbf{r_1}} + m_2\ddot{\mathbf{r_2}} = 0$$

$$\Rightarrow \ddot{\mathbf{r_2}} - \ddot{\mathbf{r_1}} = -G\frac{m_1 + m_2}{r^3}(\mathbf{r_2} - \mathbf{r_1})$$

$$\Rightarrow \ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}$$

, where $M = (m_1 + m_2)$.
Now consider the energy of the system:
The kinetic energy is given by:

$$T = \frac{1}{2}m_1|\dot{\mathbf{r_1}}|^2 + \frac{1}{2}m_2|\dot{\mathbf{r_2}}|^2$$

$$\Rightarrow T = \frac{1}{2}m_1(\frac{\mu}{m_1})^2\dot{\mathbf{r}}^2 + \frac{1}{2}m_2(\frac{\mu}{m_2})^2\dot{\mathbf{r}}^2$$

$$\Rightarrow T = \frac{1}{2}\mu\dot{\mathbf{r}}^2$$

The potential energy due to gravitational attraction is:

$$U = -G\frac{m_1 m_2}{r} = -G\frac{M\mu}{r}$$

Thus, the total energy of the system is:

$$E = T + U = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - G\frac{M\mu}{r}$$

Now on to angular momentum of the system:

$$\mathbf{L} = m_1\mathbf{r_1} \times \dot{\mathbf{r_1}} + m_2\mathbf{r_2} \times \dot{\mathbf{r_2}}$$

$$\mathbf{L} = m_1\mathbf{r_1} \times \dot{\mathbf{r_1}} + m_2\mathbf{r_2} \times \dot{\mathbf{r_2}}$$

$$\Rightarrow \mathbf{L} = m_1(-\frac{\mu}{m_1}\mathbf{r}) \times (-\frac{\mu}{m_1}\dot{\mathbf{r}}) + m_2(\frac{\mu}{m_2}\mathbf{r}) \times (\frac{\mu}{m_2}\dot{\mathbf{r}})$$

Simplifying:

$$\mathbf{L} = \mu\mathbf{r} \times \dot{\mathbf{r}}$$

Effectively, we have reduced the two-body problem to a single-body problem of mass μ moving due to the gravitational influence of a mass M fixed at origin.

3 Solving the Equations of motion

We have the equations of motion for the reduced single-body problem:

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}$$

There exist quite a few methods to tackle this differential equation. All these methods work on the same underlying principles with slightly varying pedagogical approaches. The one presented in Kleppner is quite long and uninspiring. To directly solve this problem without delving into components of the radial vector we should talk about the degrees of freedom of this problem. Since specifying the position and velocity vector of a particle completely specifies its motion, we have two 3-dimensional vectors and thus six degrees of freedom at first glance. However, we can find some constants of motion that greatly reduce this number.

3.1 Constants of Motion/Constraints of Motion

First we go back to the angular momentum of the system:

$$\mathbf{L} = \mu\mathbf{r} \times \dot{\mathbf{r}}$$

Define

$$\mathbf{k} = \frac{\mathbf{L}}{\mu} = \mathbf{r} \times \dot{\mathbf{r}}$$

Now taking the time derivative of $\mathbf{k}$:

$$\dot{\mathbf{k}} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times (-\frac{GM}{r^3}\mathbf{r}) = -\frac{GM}{r^3}\mathbf{r} \times \mathbf{r} = \mathbf{0}$$

Thus, $\mathbf{k}$ is a constant vector. Since $\mathbf{k}$ is constant, the motion of the system is constrained to a plane perpendicular to $\mathbf{k}$.
Next, consider the total energy of the system:

$$E = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{GM\mu}{r}$$

Define

$$h = \frac{E}{\mu} = \frac{1}{2}\dot{\mathbf{r}}^2 - \frac{GM}{r}$$

Now taking the time derivative of h:

$$\dot{h} = \frac{1}{2} \cdot 2\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} - GM\frac{d}{dt}(\frac{1}{r})$$

$$\Rightarrow \dot{h} = \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \frac{GM}{r^2}\frac{dr}{dt}$$

$$\Rightarrow \dot{h} = \dot{\mathbf{r}} \cdot (-\frac{GM}{r^3}\mathbf{r}) + \frac{GM}{r^2}\frac{dr}{dt}$$

Consider that $\frac{dr}{dt} = \hat{\mathbf{r}} \cdot \dot{\mathbf{r}}$, where $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$ is the unit radial vector. Thus, we have:

$$\dot{h} = -\frac{GM}{r^3}\dot{\mathbf{r}} \cdot \mathbf{r} + \frac{GM}{r^2}\mathbf{r} \cdot \dot{\mathbf{r}} = 0$$

This implies that h is a constant of motion.

The next constant is rather non-trivial and much more interesting.

3.2 The Laplace-Runge-Lenz Vector/ Eccentricity Vector

We have the expression for two vectors in terms of velocity and position; $\mathbf{k}$ and $\ddot{\mathbf{r}}$. Lets see their cross product:

$$\mathbf{k} \times \ddot{\mathbf{r}} = (\mathbf{r} \times \dot{\mathbf{r}}) \times \ddot{\mathbf{r}}$$

Using the vector triple product identity:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

we get:

$$\mathbf{k} \times \ddot{\mathbf{r}} = (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})\mathbf{r} - (\mathbf{r} \cdot \ddot{\mathbf{r}})\dot{\mathbf{r}}$$

Substituting $\ddot{\mathbf{r}} = -\dfrac{GM}{r^3}\mathbf{r}$:

$$\mathbf{k} \times \ddot{\mathbf{r}} = -(\dot{\mathbf{r}} \cdot (-\frac{GM}{r^3}\mathbf{r}))\mathbf{r} + (\mathbf{r} \cdot (-\frac{GM}{r^3}\mathbf{r}))\dot{\mathbf{r}}$$

Simplifying:

$$\mathbf{k} \times \ddot{\mathbf{r}} = \frac{GM}{r^3}(\dot{\mathbf{r}} \cdot \mathbf{r})\mathbf{r} - \frac{GM}{r}\dot{\mathbf{r}}$$

Consider that $\frac{dr}{dt} = \hat{\mathbf{r}} \cdot \dot{\mathbf{r}}$

$$\mathbf{k} \times \ddot{\mathbf{r}} = \frac{GM}{r^2}\dot{r}\mathbf{r} - \frac{GM}{r}\dot{\mathbf{r}} = -GM(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\mathbf{r}}{r^2}) = -GM\frac{d}{dt}(\frac{\mathbf{r}}{r})$$

Since we know that $\mathbf{k}$ is a constant vector, we can write the left hand side as a total time derivative:

$$\frac{d}{dt}(\mathbf{k} \times \dot{\mathbf{r}}) = \mathbf{k} \times \ddot{\mathbf{r}}$$

$$\Rightarrow \frac{d}{dt}(\mathbf{k} \times \dot{\mathbf{r}} + \frac{GM\mathbf{r}}{r}) = 0$$

Thus we get a new constant of motion:

$$\mathbf{k} \times \dot{\mathbf{r}} + \frac{GM\mathbf{r}}{r} = -GM\mathbf{e}$$

$$\mathbf{e} = -\frac{\mathbf{k} \times \dot{\mathbf{r}}}{GM} - \hat{\mathbf{r}}$$

This vector is defined so as to make it dimensionless. Note that it is formed by addition of two vectors, the former, being perpendicular to $\mathbf{k}$, lies in the plane of motion; the second trivially follows. This implies this vector lies in the orbital plane. In this form, it is known as the eccentricity vector. However it is commonly scaled up to form the Laplace-Runge-Lenz vector:

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - GM\mu^2\hat{\mathbf{r}}$$

Now, we have two vectors and one scalar as constants of motion. Somehow, we started with six degrees of freedom and lost seven. That can't be right. Turns out all three of these constants are related to each other.

3.3 Relation between Constraints

For the first relation, we can see $\mathbf{k} \cdot \mathbf{e} = -\hat{\mathbf{r}} \cdot \mathbf{k} = 0$. This implies $\mathbf{k}$ can only have two components independent of $\mathbf{e}$, in the plane perpendicular to it.
Now consider:

$$e^2 = \frac{(\mathbf{k} \times \dot{\mathbf{r}}) \cdot (\mathbf{k} \times \dot{\mathbf{r}})}{G^2 M^2} + 1 + 2\frac{(\mathbf{k} \times \dot{\mathbf{r}}) \cdot \hat{\mathbf{r}}}{GM}$$

Since $\mathbf{k}$ is perpendicular to $\dot{\mathbf{r}}$, we have:

$$e^2 = \frac{k^2 \dot{r}^2}{G^2 M^2} + 1 + 2\frac{(\dot{\mathbf{r}} \times \hat{\mathbf{r}}) \cdot \mathbf{k}}{GM} = \frac{k^2 \dot{r}^2}{G^2 M^2} + 1 - 2\frac{k^2}{GMr} = 1 + \frac{2k^2}{G^2 M^2}\left(\frac{1}{2}\dot{r}^2 - \frac{GM}{r}\right)$$

$$\Rightarrow e^2 = 1 + \frac{2k^2}{G^2 M^2} h$$

$$\Rightarrow h = \frac{G^2 M^2}{2k^2}(e^2 - 1)$$

This implies that h is completely determined by the values of e and k. This means we have effectively lost five degrees of freedom, not seven. While this is a good sign, clearly one must be left. What could this last all elusive parameter be? Of course, using these 5 constraints, we have only determined the shape and orientation of the orbit. We remain clueless on where exactly the mass μ lies on this determined orbit. To completely determine where the body is at any given time, we need its exact position at some time or simply the time where it was in some given direction. We have discussed a lot about characterising the motion of the body, now lets see how we can actually solve the equation of motion.

3.4 Solution

Since $\mathbf{e}$ lies in the orbital plane, we choose it as the reference direction. The angle between $\mathbf{e}$ and $\mathbf{r}$ is denoted by θ and referred to as the 'true anomaly'.

$$\mathbf{r} \cdot \mathbf{e} = re\cos\theta$$

But we also have:

$$\mathbf{r} \cdot \mathbf{e} = -\mathbf{r} \cdot \frac{\mathbf{k} \times \dot{\mathbf{r}}}{GM} - \mathbf{r} \cdot \hat{\mathbf{r}} = \frac{\mathbf{k} \cdot (\mathbf{r} \times \dot{\mathbf{r}})}{GM} - r = \frac{k^2}{GM} - r$$

$$\Rightarrow r = \frac{k^2}{GM(1 + e\cos\theta)}$$

This is the general equation of a conic section in polar coordinates. The eccentricity of the conic section is given by e. If $e = 0$, we have a circle; if $e < 1$, we have an ellipse; if $e = 1$, we have a parabola; and if $e > 1$, we have a hyperbola. {If you are not familiar with the general equation of a conic in polar coordinates, convert this equation into cartesian for better clarity.}
Thus we have also proved Kepler's first law.

3.5 Orbit Parameters and Kepler's Second and Third Laws

Since we now have the general conic equation, we also have the values of the semi latus rectum p and the semi major axis a. By comparing with the standard conic equation;

$$r = \frac{p}{1 + e\cos\theta}$$

we can identify:

$$p = \frac{k^2}{GM}$$

and

$$a = \frac{p}{|1 - e^2|} = \frac{k^2}{GM|1 - e^2|}$$

Clearly, we have $a = -\dfrac{GM}{2h}$ for an elliptical/circular orbit and $a = \dfrac{GM}{2h}$ for a hyperbolic orbit. For a parabolic orbit, $a \to \infty$.
Also, r is the least at $\theta = 0$, i.e. pointing along the eccentricity vector, and is the greatest at $\theta = \pi$, i.e. pointing opposite to the eccentricity vector. The distance at $\theta = 0$ is called the periapsis r_p

and at $\theta = \pi$ is called the apoapsis r_a.
Homework problem(I'm too lazy to type the solution): Derive the vis-viva equation for any general orbit:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

The areal velocity of the body, i.e., the area swept per unit time, upto first order, is given by:

$$\dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}r \cdot r\dot{\theta} = \frac{1}{2}|\mathbf{r} \times \dot{\mathbf{r}}| = \frac{1}{2}k$$

This implies areal velocity is constant for a general orbit, thus proving Kepler's second law.
From here on out, we will focus on elliptical orbits for the sake of approachability, however, please acknowledge that the problem of orbital motion is completely solvable for both parabolic and hyperbolic orbits. Integrating the previous result over an entire orbit will yield:

$$\pi a^2\sqrt{1-e^2} = \frac{1}{2}kP$$

Simplifying;

$$P^2 = \frac{4\pi^2 a^3}{GM}$$

Thus we have derived Kepler's third law.

3.6 Kepler's Equation

While the trajectory in the plane of motion makes for a neat solution, it is more interesting to parametrize the last degree of freedom using time to get a truly complete picture of the orbit. Consider the the auxiliary circle of the elliptical orbit. Then the 'eccentric anomaly', E is the angle of the 'scaled position' measured from the centre of the circle.

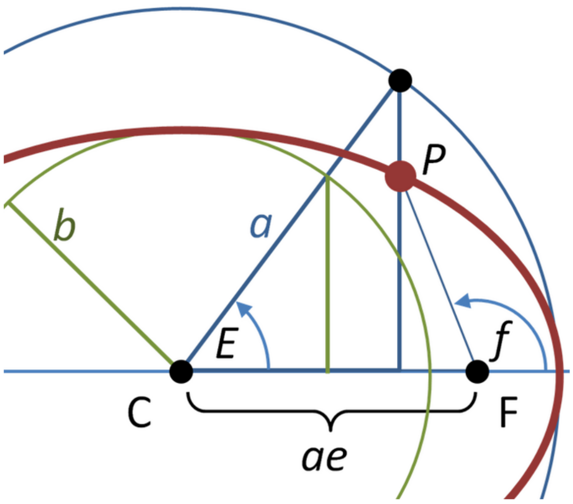

Figure 1: Illustration of the eccentric anomaly E in the auxiliary circle of the elliptical orbit.

In terms of E, the cartesian coordinates become rather trivial:

$$x = a(\cos E - e), \quad y = b\sin E$$

where $b = a\sqrt{1-e^2}$ is the semi-minor axis.
So our problem is reduced to finding E as a function of time. This is done using the concept of the 'mean anomaly', M', which is defined as the angle that the radius vector would sweep out if the body moved at a constant angular speed.

$$M' = \frac{2\pi}{P}t$$

where t is the time since periapsis passage.
Since the areal velocity is constant, the area swept out in this time t is:

$$A = \pi ab\frac{t}{P} = \frac{M'ab}{2}$$

Another way to write this angle is as the scaled area of a 'sector' measured from the focus:

$$A = \frac{b}{a}(\frac{a^2 E}{2} - \frac{ae \cdot a \cdot \sin E}{2}) = \frac{ab}{2}(E - e\sin E)$$

Comparing the two equations, we get the Kepler equation, the complete solution of the problem we started with:

$$M' = E - e\sin E$$

This is a transcendental equation and cannot be solved analytically using elementary functions. However, it can be solved numerically using many methods one of which is Newton's method. While complete, the solution is not exactly trivial and is not, for the lack of a better term, awe inspiring. While the usual position space is cursed with such complexities, something much more elegant turns up in the velocity space.

4 Hodographs

Hodographs are graphs used to give a visual representation of the movement of a body or fluid. The variable vector being studied is plotted in its space, with the tail fixed at origin. While dealing with velocity vectors in a Keplerian orbit we find something interesting. Lets look again at the definition of the eccentricity vector:

$$\mathbf{e} = -\frac{\mathbf{k} \times \dot{\mathbf{r}}}{GM} - \hat{\mathbf{r}}$$

To isolate the velocity, we can cross multiply with $\mathbf{k}$:

$$\mathbf{k} \times \mathbf{e} = -\mathbf{k} \times \left(\frac{\mathbf{k} \times \dot{\mathbf{r}}}{GM}\right) - \mathbf{k} \times \hat{\mathbf{r}}$$

$$\Rightarrow \mathbf{k} \times \mathbf{e} = -\frac{(\mathbf{k} \cdot \dot{\mathbf{r}})\mathbf{k} - k^2 \dot{\mathbf{r}}}{GM} - \mathbf{k} \times \hat{\mathbf{r}} = \frac{k^2}{GM}\dot{\mathbf{r}} - \mathbf{k} \times \hat{\mathbf{r}}$$

Rearranging, we get:

$$\dot{\mathbf{r}} = \frac{GM}{k^2}(\mathbf{k} \times \mathbf{e}) + \frac{GM}{k^2}(\mathbf{k} \times \hat{\mathbf{r}})$$

The first term of the expression is a constant vector and the magnitude of the second term is $|\frac{GM}{k^2}(\mathbf{k} \times \hat{\mathbf{r}})| = \frac{GM}{k^2}|\mathbf{k} \times \hat{\mathbf{r}}| = \frac{GM}{k^2}|\hat{\mathbf{r}}| = \frac{GM}{k}$, which is a constant. Rearranging:

$$|\dot{\mathbf{r}} - \frac{GM}{k^2}(\mathbf{k} \times \mathbf{e})| = \frac{GM}{k}$$

This is the equation of a circle, with radius $\frac{GM}{k}$ and centre at $\frac{GM}{k^2}(\mathbf{k} \times \mathbf{e})$. The velocity vector traces out a circle in the hodograph space, with the centre and radius determined by the constants of motion, clearly demonstrating the motion as one with a single degree of freedom.
Homework problem: Solve Physics Cup 2024 Problem 4.

5 A Brief Discussion on Three Body Problem

While we solved the problem of two bodies bound gravitationally by reducing it to a single body problem and finding an analytical solution, the three body problem is not so easily solvable. The equations of motion for three bodies are given by:

$$\ddot{\mathbf{r}}_1 = -\frac{GM_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\hat{\mathbf{r}}_{12} - \frac{GM_3}{|\mathbf{r}_1 - \mathbf{r}_3|^2}\hat{\mathbf{r}}_{13}$$

$$\ddot{\mathbf{r}}_2 = -\frac{GM_1}{|\mathbf{r}_2 - \mathbf{r}_1|^2}\hat{\mathbf{r}}_{21} - \frac{GM_3}{|\mathbf{r}_2 - \mathbf{r}_3|^2}\hat{\mathbf{r}}_{23}$$

$$\ddot{\mathbf{r}}_3 = -\frac{GM_1}{|\mathbf{r}_3 - \mathbf{r}_1|^2}\hat{\mathbf{r}}_{31} - \frac{GM_2}{|\mathbf{r}_3 - \mathbf{r}_2|^2}\hat{\mathbf{r}}_{32}$$

Newton is said to have spent sleepless nights trying to solve this problem, but to no avail. Turns out a closed form solution for it doesn't exist. (An analytical solution does exist, but is so complicated that for any reasonable precision you need to evaluate $10^{8000000}$ terms.) Numerically evaluating these equations is possible, but the system is chaotic, meaning that even a small change in initial conditions can lead to vastly different outcomes. This makes it difficult to predict the long-term behavior of the system. There are some periodic solutions of the three body problem that are mostly predictable; however, they require further constraints on the masses of the three bodies. The most commonly known periodic solutions assume the two bodies in orbit and a third, considered massless with respect to the other two. Three of these were first discovered by Euler (because of course they were). These are points collinear with the two main bodies at all times, i.e., with the same angular velocity about the centre of mass. Two were later discovered by Lagrange, which are the two points that form an equilateral triangle with the two main bodies at all times. These five points are the Lagrange points.

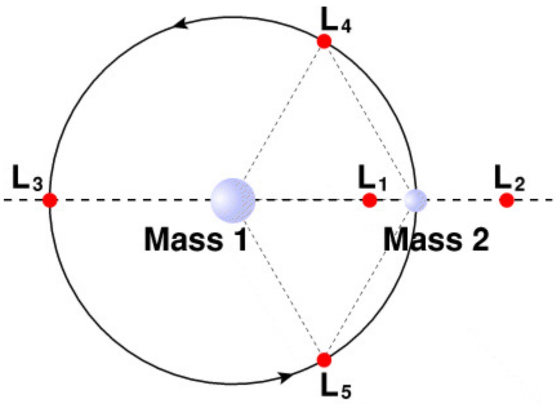

Figure 2: Illustration of the Lagrange points in the three-body problem.

Homework problem: Given what you know till now, find the exact location of the Lagrange points

$L1$, $L2$ and $L3$ are considered meta-stable with respect to small perturbations, while $L4$ and $L5$ are stable. Lots of asteroids and other small bodies tend to collect in or near the Lagrange points of the planets, particularly Jupiter.

6 Many-Body Problem and Virial Theorem

Chaos reared its head in as little as three bodies, it is suffices to say that a system of large number of bodies gravitationally bound to each other is anything but predictable. However we can do some basic analysis to find a rather nice property, the Virial Theorem. Consider a system of n bodies with positions $\mathbf{r}_i$ and masses m_i and velocities $\dot{\mathbf{r}}_i$. We define a quantity A, called the virial, as:

$$A = \sum_{i=1}^{n} m_i \mathbf{r}_i \cdot \dot{\mathbf{r}}_i$$

Taking its time derivative:

$$\frac{dA}{dt} = \sum_{i=1}^{n} m_i \left(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i + \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i \right)$$

The first term is twice the kinetic energy of the i-th body, while the second term involves the gravitational forces acting on the bodies.

$$\Rightarrow \frac{dA}{dt} = 2T + \sum_{i=1}^{n} \mathbf{r}_i \cdot \mathbf{F}_i$$

Averaging over an interval τ gives:

$$\frac{1}{\tau} \int_0^{\tau} \frac{dA}{dt} dt = \frac{1}{\tau} \left(A(\tau) - A(0) \right) = \langle 2T \rangle + \langle \sum_{i=1}^{n} \mathbf{r}_i \cdot \mathbf{F}_i \rangle$$

If the system remains bounded over a long time we can consider the left hand side to be zero.

$$\Rightarrow \langle 2T \rangle + \langle \sum_{i=1}^{n} \mathbf{r}_i \cdot \mathbf{F}_i \rangle = 0$$

This is the general form of the virial theorem. Since we are only concerned with mutual gravitation;

$$\mathbf{F}_i = -Gm_i \sum_{j=1, j\neq i}^{n} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}^3}$$

The latter term in the virial theorem is now

$$\sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{r}_i = -G \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{m_i m_j}{r_{ij}} (\mathbf{r}_i - \mathbf{r}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)$$

where the latter form is obtained by rearranging the double sum, combining the terms

$$m_i m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}^3} \cdot \mathbf{r}_i$$

and

$$m_j m_i \frac{\mathbf{r}_j - \mathbf{r}_i}{r_{ji}^3} \cdot \mathbf{r}_j = m_i m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}^3} \cdot (-\mathbf{r}_j)$$

Since $(\mathbf{r}_i - \mathbf{r}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) = r_{ij}^2$ the sum reduces to

$$-G \sum_{i=1}^{n} \sum_{j=1, j>i}^{n} \frac{m_i m_j}{r_{ij}} = U$$

where U is the potential energy of the system. Thus, the virial theorem becomes simply

$$\langle T \rangle = -\frac{1}{2} \langle U \rangle$$

7 Binary Stars

Binary star systems are some of the most common systems you'll observe in the night sky. They consist of two stars orbiting around their common center of mass. The study of binary stars is crucial for deetermining stellar masses, luminosities, and evolutionary stages.

7.1 Types of Binary Stars

- **Optical Doubles:** These are two stars which are not physically associated but appear close together in the sky. They are not gravitationally bound and do not orbit each other. Also referred to as false binaries.
- **Visual Binaries:** These are binary systems where the two stars can be resolved as separate objects through a telescope. It is often possible to monitor their orbital motion around each other. The most common example is the binary star system Alpha Centauri, which consists of three stars: Alpha Centauri A, Alpha Centauri B, and Proxima Centauri.
- **Astrometric Binaries:** In these systems, one star is too faint to be observed directly, but its presence is inferred from the gravitational influence it exerts on a brighter companion star, causing it to wobble in its motion. The same principle is often used to find extrasolar planets. A well-known example is the star Sirius, which has a faint companion, Sirius B.
- **Spectroscopic Binaries:** In these systems, the stars are too close together to be resolved individually, but their presence is inferred from the Doppler shifts in their spectral lines, which changes as they move about each other. For an example, the binary system Beta Persei (Algol) is a spectroscopic binary.
- **Eclipsing Binaries:** These are systems where the orbital plane is aligned with our line of sight, causing the stars to periodically eclipse each other. This results in variations in luminosities which are easily measured and reveal a lot more than just the presence of two stars. A famous example is the binary star system Algol, which exhibits regular eclipses every 2.87 days. Yes, the Algol system has three confirmed stars, two are an eclipsing binary while the third was only discovered by spectroscopic measurements.

7.2 Mass Determination using Visual Binaries

In visual binaries, the orbital motion of the two stars can be observed directly. By measuring the period of their orbit and the semi-major axis of their orbit, we can determine their masses using Kepler's third law:

$$\frac{a^3}{P^2} = \frac{G(M_1 + M_2)}{4\pi^2}$$

where a is the semi-major axis, P is the orbital period, G is the gravitational constant, and M_1 and M_2 are the masses of the two stars. Rearranging this equation allows us to solve for the total mass of the system:

$$M_1 + M_2 = \frac{4\pi^2 a^3}{GP^2}$$

Mass ratio is readily obtained from the relative distances of the two stars from the center of mass, which can be determined from their angular separation and the semi-major axis:

$$\frac{M_1}{M_2} = \frac{a_2}{a_1}$$

where a_1 and a_2 are the semi-major axes of the two stars' orbits, and P_1 and P_2 are their respective orbital periods. All of this analysis is trivially done as the entire motion of the system is visually tracked. The only possible complication can arrive from the plane of the orbit being inclined at an angle i to the perpendicular to the line of sight. Even then, using geometry, we can infer this angle as the centre of mass would not lie on the focus of the observed ellipse.

7.3 Mass Determination using Eclipsing, Spectroscopic Binaries

In eclipsing binaries, the orbital motion of the two stars can be observed indirectly through the changes in brightness as one star passes in front of the other. By analyzing the light curves, we can determine the relative sizes and temperatures of the stars, as well as their orbital parameters. From here onwards we will consider circular orbits for simplicity, as non-zero eccentricities tend to make the light curves much more complicated. This is not too far from reality, as in most binary systems, huge tidal forces tend to circularize the orbits.

Consider two binary stars whose orbit is inclined to the perpendicular to the line of sight by an angle i. The observed radial velocities for both the stars will be a maximum of $v_1 \sin i$ and $v_2 \sin i$, where v_1 and v_2 are the actual radial velocities of the stars. The mass ratio can be determined from the ratio of the observed velocities:

$$\frac{M_1}{M_2} = \frac{v_2 \sin i}{v_1 \sin i} = \frac{v_2}{v_1}$$

So, the mass ratio can be determined accurately from spectroscopic data regardless of the inclination angle. For the semi major axis a we can use the fact that the orbit is circular to get:

$$a = a_1 + a_2 = \frac{P}{2\pi}(v_1 + v_2) = \frac{P}{2\pi \sin i}(v_{1r} + v_{2r})$$

Since we now know a we can determine the total mass of the system using Kepler's third law:

$$M_1 + M_2 = \frac{4\pi^2 a^3}{GP^2} = \frac{4\pi^2}{GP^2}\left(\frac{P}{2\pi \sin i}(v_{1r} + v_{2r})\right)^3$$

$$\Rightarrow M_1 + M_2 = \frac{P}{2\pi G}\frac{(v_{1r} + v_{2r})^3}{\sin^3 i}$$

Then masses of both stars can be determined provided we know the inclination angle i. This can be estimated to a good amount by studying the 'flatness' of the minimum of the light curves of the system. This analysis is done assuming we know the radial velocity of both stars by spectroscopic analysis. However, this is not always the case, as often, one of the two stars in such a system is significantly brighter than the other, overwhelming the spectroscopic signals from the fainter star. In such a case, the best we can do is:

$$M_1 + M_2 = \frac{P}{2\pi G}\frac{(M_2 + M_1)^3 v_{1r}^3}{M_2^3 \sin^3 i}$$

$$\Rightarrow \frac{M_2^3}{(M_1+M_2)^2}\sin^3 i = \frac{P}{2\pi G}v_{1r}^3$$

The right hand side, known as the mass function, can be completely determined by observational data. If inclination is not known the masses cannot be determined. However, using an estimate of the mass of the brighter star using standard luminosity-mass relations, we can use mass function to obtain a lower bound for the mass of the fainter one. Surface temperature ratios and approximate radii of the stars can also be determined relatively accurately from the luminosity time curve.

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